A probabilistic study on the mixed-mode fracture in functionally graded materials

H. Z. Pan[1*]

[1]College of Computer and Control Engineering, Qiqihar University, Qiqihar, China

*Corresponding author: haizhupan@163.com

1. Introduction

The main feature of functionally graded materials (FGMs) is spatially varying microstructures and macroproperties. It makes the FGMs can be more appropriately applied to some extremely special environments to evaluate the mechanical reliability of FGMs and understand their deformation and fracture behavior. However the effects of the material microstructures on the mechanical properties of FGMs embedded the crack should not be neglected. Therefore, it is essential to study the fracture characteristics of FGMs from a microstructures perspective. Some significant efforts have been made in the study of the fracture behavior of FGMs embedded the crack with respect to microstructures of the material.

Some scholars investigated the uncertainties in material properties of FGMs. Rahman and Chakraborty presented a stochastic micromechanical model to forecast probabilistic characteristics of elastic mechanical properties of FGMs [1]. A probabilistic fracture analysis framework was proposed to study the effects of uncertainties on the fracture responses of FGM structures by Song et al. [2]. Xu et al. [3] investigated the effects of random constituent material properties on the dynamic characteristics for FGM beam by the random factor method. Ferrante and Graham-Brady [4] investigated the inherently random nature of graded structures with respect to the performance of the component for a thermal barrier application. And their results showed that deviations in ceramic/metal volume fraction would produce significant randomness in the stress. Ilschner [5] discussed the complex relationships of the macroscopic properties of functionally graded components, the local composition function and the microstructural parameters (porosity, grain size). The foregoing archives are based on the randomnees of FGMs microstructure. They laid the foundation for further study of the uncertain fracture response of FGMs with cracks. Chakraborty and Rahman [6] analyzed a functionally graded specimen with an edge crack under a mixed-mode deformation by a numerical method to calculate the statistical moments of crack-driving forces and the probability of fracture initiation. Taking into account its microstructural properties, Wu and Du [7] obtained the effective elastic moduli of an inhomogeneous medium with cracks. Nguyen et al. [8] estimated the probability of crack initiation with uncertainties in the material properties by the computational simulation. A numerical technique to model the effect of uncertainties in the crack geometry on the reliability of cracked structures was presented by Chowdhury et al.[9]. Lal et al.[10] applied a stochastic extended finite element method to the fracture analysis of central cracked laminated composite plate under uni-axial tension with random system properties. Yang et al. [11] studied the bending responses of thermo-mechanically loaded compositionally graded plates with randomness in material properties and volume composition. Lal et al. [12] provided a probabilistic tool for incorporating and handling the structural material uncertainties in the analysis of the structures. In previous investigations, only a few of them analyzed analytically the crack problems of FGMs considering the stochastic mechanical properties. Guo et al. [13] developed the analytical model for these problems. But they only considered mode-I crack problem in it. Zhang et al. proposed an analytical approach for the random dynamic analysis of a functionally graded material layer containing a crack between two dissimilar elastic half-planes. But they considered mode-III crack problem under transient loadings. Less relevant researches on the influences of the randomness of microstructural properties on mixed-mode stress intensity factors (SIFs) of FGMs can be found.

In this paper, to investigate effects of random variations in the component volume fractions on SIFs of FGMs with an internal slant crack subjected to external loadings, an efficient analytical solution for modes I and II SIFs of multiple specimens with respect to randomness of shear modulus is presented. And then based upon the results, probability density function diagrams and specimen distribution diagrams of modes I and II SIFs are drawn, which can be illustrate the effects of the randomness of the meso structure of the FGM strip on the SIFs.

2. Body of abstract

This article reports the results of a probabilistic study on the mixed-mode fracture problem of an internal slant crack in a functionally graded material strip. The study involves randomness description of the micro-structural attributes of functionally graded materials; the effects of the random micro-structural attributes on the macroscopic properties of functionally graded materials; analytical solutions of modes I and II stress intensity factors and analysis of its probabilistic characteristics. The influences of the randomness in micro-structural component on the statistics (e.g. the mean, the standard deviation) of the shear modulus of functionally graded materials are graphically represented. The results also reveal this research may be more meaningful on the probabilistic characteristic of stress intensity factors of functionally graded materials with the larger size slant crack.

Keywords：Probabilistic characteristics; Micro-structural attributes; Randomness; Stress intensity factors; Functionally graded materials

3. Equations, figures, and tables

Equations in this paper are as follows.

$$\mu(x) = \zeta\left(x\right) \tag{1}$$

$$\mu(x) = \mu_{n0} e^{\beta_n x},\ n = 1,2,...,m \tag{2}$$

$$\mu(x_1, y_1) = \mu_{n0} e^{\beta_n(\cos\theta x_1 - \sin\theta y_1)},\ n = 1,2,...,m \tag{3}$$

$$\mu(h_n) = \zeta(h_n) = \mu_{n0} e^{\beta_n h_n} \text{ and } \mu(h_{n-1}) = \zeta(h_{n-1}) = \mu_{n0} e^{\beta_n h_{n-1}} \tag{4}$$

$$\mu_{n0} = \zeta(h_n) e^{-\delta_n h_n} \text{ and } \delta_n = \frac{\ln\left[\zeta(h_n)/\zeta(h_{n-1})\right]}{h_n - h_{n-1}} \tag{5}$$

$$\begin{cases} \sigma_{nxx} = \dfrac{\mu_n(x)}{\kappa_n(x)-1}\left\{\left[1+\kappa_n(x)\right]\dfrac{\partial u_n}{\partial x} + \left[3-\kappa_n(x)\right]\dfrac{\partial v_n}{\partial y}\right\} \\ \sigma_{nyy} = \dfrac{\mu_n(x)}{\kappa_n(x)-1}\left\{\left[1+\kappa_n(x)\right]\dfrac{\partial v_n}{\partial y} + \left[3-\kappa_n(x)\right]\dfrac{\partial u_n}{\partial x}\right\},\ n=1,2,...,m \\ \tau_{nxy} = \mu_n(x)\left(\dfrac{\partial u_n}{\partial y} + \dfrac{\partial v_n}{\partial x}\right) \end{cases} \tag{6}$$

$$\begin{cases} \dfrac{\partial \sigma_{nxx}}{\partial x} + \dfrac{\partial \tau_{nxy}}{\partial y} = 0 \\ \dfrac{\partial \sigma_{nyy}}{\partial y} + \dfrac{\partial \tau_{nxy}}{\partial x} = 0 \end{cases},\ n = 1,2,...,m \tag{7}$$

$$\sigma_{1xx}(0, y) = 0 \text{ and } \tau_{1xy}(0, y) = 0,\ -\infty < y < \infty \tag{8}$$

$$\sigma_{mxx}(h, y) = 0 \text{ and } \tau_{mxy}(h, y) = 0, -\infty < y < \infty \tag{9}$$

$$\sigma_{ny_1y_1}(x_1, 0) = f_1(x_1) \text{ and } \tau_{nx_1y_1}(x_1, 0) = f_2(x_1),\ a < x_1 < b \tag{10}$$

$$\sigma_{nxx}\left(h_n, y\right) = \sigma_{(n+1)xx}\left(h_n, y\right) \text{ and } \tau_{nxy}\left(h_n, y\right) = \tau_{(n+1)xy}\left(h_n, y\right),\ n = 1,2,...,m-1 \tag{11}$$

$$u_n\left(h_n, y\right) = u_{n+1}\left(h_n, y\right) \text{ and }$$

$$v_n\left(h_n, y\right) = v_{n+1}\left(h_n, y\right), n = 1,2,...,m-1 \tag{12}$$

$$\begin{cases} \lim\limits_{y_1\to 0^+}[\sigma^I_{ny_1y_1} + \sigma^{II}_{nxx}\sin^2\theta + \sigma^{II}_{nyy}\cos^2\theta - 2\tau^{II}_{nxy}\sin\theta\cos\theta] = f_1(x_1) \\ \lim\limits_{y_1\to 0^+}[\tau^I_{ny_1y_1} - \sigma^{II}_{nxx}\sin\theta\cos\theta + \sigma^{II}_{nyy}\sin\theta\cos\theta + \tau^{II}_{nxy}(\cos^2\theta - \sin^2\theta)] = f_2(x_1) \end{cases},\ a < x_1 < b \tag{13}$$

$$K_I(a) = \lim_{x_1\to a}\sqrt{2(a-x_1)}\sigma_{y_1y_1}(x_1,0) = -\frac{2\sqrt{2}\mu_{n0}}{\kappa+1}e^{\beta_n a}\frac{w_2(a)}{\sqrt{c}} \tag{14}$$

$$K_I(b) = \lim_{x_1\to b}\sqrt{2(x_1-b)}\sigma_{y_1y_1}(x_1,0) = \frac{2\sqrt{2}\mu_{n0}}{\kappa+1}e^{\beta_n b}\frac{w_2(b)}{\sqrt{c}} \tag{15}$$

$$K_{II}(a) = \lim_{x_1\to a}\sqrt{2(a-x_1)}\tau_{x_1y_1}(x_1,0) = -\frac{2\sqrt{2}\mu_{n0}}{\kappa+1}e^{\beta_n a}\frac{w_1(a)}{\sqrt{c}} \tag{16}$$

$$K_{II}(b) = \lim_{x_1\to b}\sqrt{2(x_1-b)}\tau_{x_1y_1}(x_1,0) = \frac{2\sqrt{2}\mu_{n0}}{\kappa+1}e^{\beta_n b}\frac{w_1(b)}{\sqrt{c}} \tag{17}$$

$$E_{V_p}(x) = \sum\nolimits_{j=0}^{5} p_{ij}x^j \tag{18}$$

$$S_{V_p}(x) = S_{V_0} = \sum\nolimits_{j=0}^{5} q_{ij}x^j \tag{19}$$

$$\mu(x) = \mu_m + (\mu_p - \mu_m)V_p(x)/\{1 + 4[1 - V_p(x)]\overline{\mu_p}(\mu_p - \mu_m)\} \tag{20}$$

$$\overline{\mu_p} = 3(2\mu_m + k_m)/[10\mu_m(4\mu_m + 3k_m)] \tag{21}$$

Figures and tables in this paper are as follows.

Table 1 The mean and standard deviation of SIFs for crack samples with different angle when $\mu_p / \mu_m = 20$, $S_{Vp} = 2S_{V0}$ and $c/h = 0.4$

Crack angle	Mean of $K_I(a)/K_0$	Standard deviation of $K_I(a)/K_0$	Mean of $K_I(b)/K_0$	Standard deviation of $K_I(b)/K_0$
0. 1 π	0. 9803	0. 0063	1. 0861	0. 0041
0. 15 π	0. 8630	0. 0050	0. 9608	0. 0036
0. 2 π	0. 7135	0. 0036	0. 8003	0. 0030
0. 25 π	0. 5458	0. 0024	0. 6194	0. 0024
0. 3 π	0. 3762	0. 0014	0. 4351	0. 0018
0. 33 π	0. 2808	0. 0010	0. 3305	0. 0014
0. 35 π	0. 2218	0. 0007	0. 2654	0. 0012
0. 4 π	0. 0796	0. 0002	0. 1053	0. 0006
Crack angle	Mean of $K_{II}(a)/K_0$	Standard deviation of $K_{II}(a)/K_0$	Mean of $K_{II}(b)/K_0$	Standard deviation of $K_{II}(b)/K_0$
0. 1 π	0. 3036	0. 0023	0. 3099	0. 001
0. 15 π	0. 4209	0. 0028	0. 4294	0. 0013
0. 2 π	0. 4994	0. 0031	0. 5092	0. 0015
0. 25 π	0. 5307	0. 0029	0. 5401	0. 0014
0. 3 π	0. 5102	0. 0025	0. 5180	0. 0013
0. 33 π	0. 4729	0. 0021	0. 4792	0. 0011
0. 35 π	0. 4376	0. 0017	0. 4430	0. 0009
0. 4 π	0. 2923	0. 0009	0. 2946	0. 0006

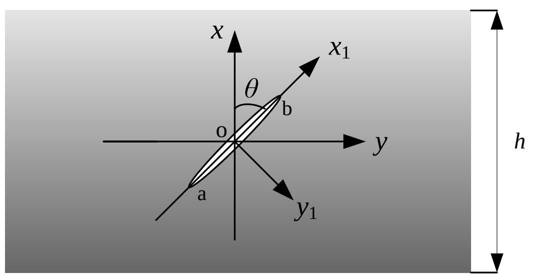

Fig.1 A slant crack embedded in a functionally

graded strip

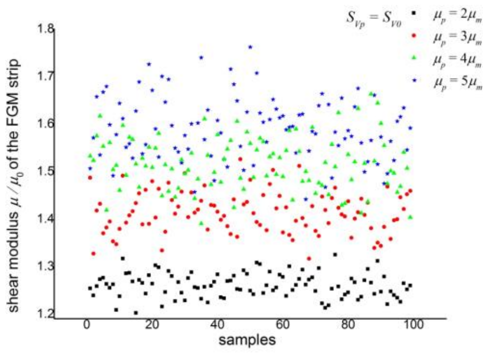

Fig.2 Distribution of samples values about the shear modulus with different modulus ratios

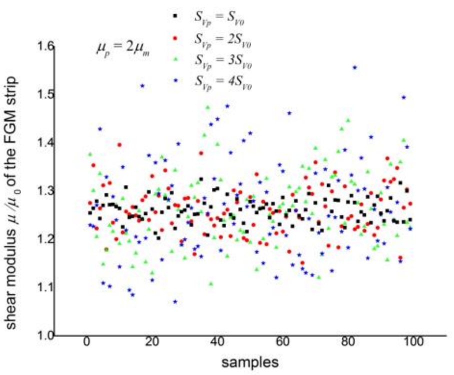

Fig.3 Distribution of samples values about the shear modulus with different standard deviations

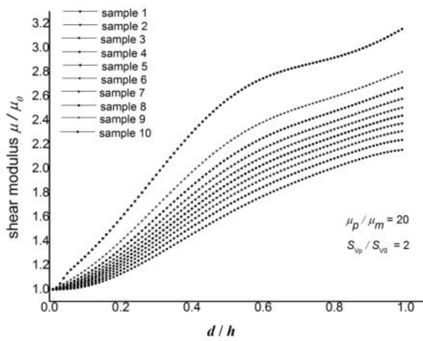

Fig.4 Different samples curves of the shear modulus (in which $d = x$+0.5)

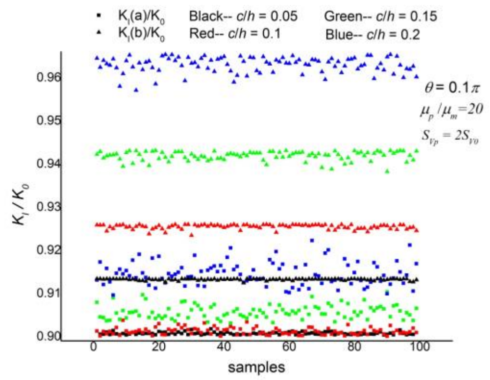

Fig.5 Distribution of mode-I SIFs for different samples with different crack length

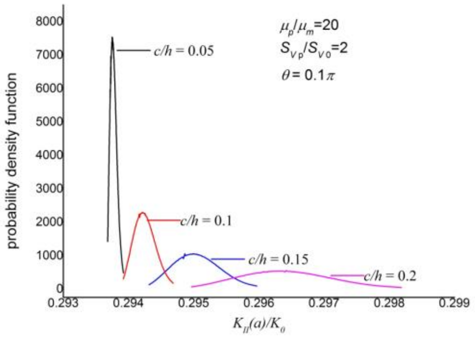

Fig.6 Probability density function of mode-II SIFs at the crack tip-*a* for different crack length

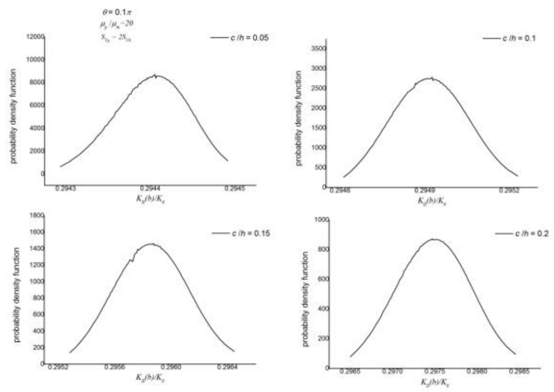

Fig.7 Probability density function of mode-II SIFs at the crack tip-*b* for different crack length

Acknowledgment

This study was funded by the Fundamental Research Funds in Heilongjiang Provincial University (135309468).

References

[1] S. Rahman and A. Chakraborty, A stochastic micromechanical model for elastic properties of functionally graded materials, *Mechanics of Materials*, 39 (2007) 548-563.

[2] J. Song, T. H. Nguyen and G. H. Paulino,

Probabilistic fracture analysis of functionally materials – Part I: uncertainty and probabilistic analysis method, *American Institute of Physics*, 973 (2008) 153-158.

[3] Y. L. Xu, Y. Qian, J. J. Chen and G. B. Song, Stochastic dynamic characteristics of FGM beams with random material properties, *Composite Structures*, 133 (2015) 585-594.

[4] F. J. Ferrante and L. L. Graham-Brady Stochastic simulation of non-Gaussian/non-stationary properties in a functionally graded plate, *Computer Methods in Applied Mechanics and Engineering*, 194 (12-16) (2005) 1675-1692.

[5] B. Ilschner, Processing-microstructure-property relationships in graded materials, *Journal of the Mechanics and Physics of Solids*, 44 (5) (1996) 647-656.

[6] A. Chakraborty and S. Rahman, A parametric study on probabilistic fracture of functionally graded composites by a concurrent multiscale method, *Probabilistic Engineering Mechanics*, 24 (2009) 438-451.

[7] L. Z. Wu and S. Y. Du, Effective elastic moduli of an inhomogeneous medium with cracks, *Acta Mechanica Sinica*, 11 (1995) 153-161.

[8] T.H. Nguyen, J. Song and G. H. Paulino, Probabilistic fracture analysis of functionally materials – Part II: Implementation and numerical examples. American Institute of Physics 973 (2008) 159-164.

[9] M. S. Chowdhury, C. M. Song and W. Gao, Probabilistic fracture mechanics by using Monte Carlo simulation and the scaled boundary finite element method, *Engineering Fracture Mechanics*, 78 (2011) 2369-2389.

[10] A. Lal, S. P. Palekar, S. B. Mulani and R. K. Kapania, Stochastic extended finite element implementation for fracture analysis of laminated composite plate with a central crack, *Aerospace Science and Technology*, 60 (2017) 131-151.

[11] J. Yang, K. M. Liew and S. Kitipornchai, Stochastic analysis of compositionally graded paltes with system randomness under static loading, *International Journal of Mechanical Sciences*, 47 (2005) 1519-1541.

[12] A. Lal, H. N. Singh and N. L. Shegokar, FEM model for stochastic mechanical and thermal postbuckling response of functionally graded material plates applied to panels with circular and square holes having material randomness, *International Journal of Mechanical Sciences*, 62 (1) (2012) 18-33.

[13] L. C. Guo, Z. H. Wang and N. Noda, A fracture mechanics model for a crack problem of functionally graded materials with stochastic mechanical properties, *Proceedings of the Royal Society A-Mathematical Physical and Sciences*, 468 (2146) (2012) 2939-2961.

[14] H. Z. Zhang, X. H. Zhao and J. X. Su, Random dynamic response and reliability of a crack in a functionally graded material layer between two dissimilar elastic half-planes. *Engineering Analysis with Boundary Elements*, 36 (2012) 1560-1570.

[15] L. C. Guo and N. Noda, Modeling method for a crack problem of functionally graded materials with arbitrary properties—piecewise-exponenti--al model, *International Journal of Solids and Structures*, 44 (2007) 6768-6790.

[16] F. Erdogan and G. D. Gupta, On the numerical solution of singular integral equations, *Quarterly of Applied Mathematics*, 29 (1972) 525-534.

[17] Z. H. Wang, L. C. Guo and L. Zhang, A general modelling method for functionally graded materials with an arbitrarily oriented crack. *Philosophical Magazine*, 94 (2014) 764-791.